Exercises Integer Linear Programming

Exercise 1
To earn more money, a student X decides to offer tutoring in mathematics for highschool students. It turns out that there are more than enough interested highschool students. X decides to take a systematic approach in planning her work. She divides her available time in T intervals of one hour. In each hour, she can do one tutoring lesson. Next, X asks every interested high-school student j ($j = 1, \ldots, n$), to indicate

- during which of the T interval he/she can take a lesson; a_{jt} equals 1 if student j can take a lesson during interval t, and 0 otherwise.
- how many lessons he/she at most wishes to take, denoted by q_j.
- which amount he/she is willing to pay for the lessons; p_{jt} denotes the amount student j is willing to pay for a lesson during interval t.

Clearly, X wants to maximize the amount she earns.

a) Formulate the above problem as an integer linear programming problem. Give a description of the decision variables, objective and constraints.

b) The high-school students do not like this procedure. Usually, just a few lessons do not help. If a high-school student takes lessons with X, he/she wants to take the full set of q_j lessons. Give an integer linear programming formulation for this situation. Give a description of the decision variables, objective and constraints.

Exercise 2
Formulate the following problem as integer linear programming problem. A company hires employees from a pool of 80 people. Employees are hired on a day to day basis. The pool contains 25 experienced persons. An experienced person can do twice as much work as an unexperienced person in the same time. Our company has n projects for which it has to recruit people. For the upcoming T days employees have to be divided over the projects. For project i ($i = 1, ..., n$) is w_{it} the amount of required work on day t ($t = 1, ..., T$); an unexperienced employee can perform one unit of work per day. If an employee is hired on a certain day, he/she works on the same project all day. The cost for hiring a normal employee are c_1 per day and the cost for hiring an experienced employee are c_2 per day. Finally, for every day the minimal number of experienced employees that has to work on project i equals u_i. The company wants to minimize the cost of hiring employeees.

Exercise 3
Consider the knapsack problem. We have n items, each with weight a_j ($j = 1, ..., n$) and value c_j ($j = 1, ..., n$) and an integer B. All a_j and c_j are positive integers. The question is to find a subset of the items with total weight at most B and maximal value.

Consider a variant of Knapsack in which it is allowed to increase B by an integral number which is at most 10. This comes with a cost, the cost for the first unit of increase equals 2, the second and third unit each cost 3, the fourth and fifth cost 6 per unit and the sixth to 10th unit cost 10 per unit. For example, increasing B by 6 costs $2 + 3 + 3 + 6 + 6 + 10 = 30$.

a) Formulate the above problem as an integer linear programming problem. Give a description of the decision variables, objective and constraints.

b) Suppose that the cost per unit increase are non-increasing, e.g. 2 for the first unit and 1 for the second unit. Check if your model is still correct. If not, give a modified integer linear programming model.

Exercise 4

This problem is about brewing beer and is based in the course Information Science course SMOI. In this course teams of students virtually run a beer company. Suppose you are the manager production and logistics in one of the teams.

- There are 7 types of beer; denoted by $b(j)$ $(j = 1, \ldots, 7)$.
- Producing beer requires Production Clusters (PCs), which can be leased. There are 6 types of PCs, denoted by $PC(i)$ $(i = 1, \ldots, 6)$. For each $PC(i)$ we are given the lease cost c_i and the production capacity a_{ij} defined as the number of crates of $b(j)$ that can be produced on $PC(i)$.
- On each cluster that you lease you can produce one type of beer. However, if you lease more than one cluster of type i, you can produce different types of beer on different clusters. There are enough production clusters available for leasing.

(a) At some point, the team has decided on the following production plan: the number of produced crates of type $b(j)$ has to be between $l(j)$ en $u(j)$; one expects to sell these crates at a price of p_j per crate. The objective is to maximize profit. Formulate the problem of optimizing the production as an integer linear programming problem. Give a description of the decision variables, objective and constraints.

(b) It is possible to produce more than $u(j)$ crates; the additional crates can be sold at a lower price r_j. Producing less than $l(j)$ is not an option. Extend the model of part (a) to include this option.

(c) For brewing beer, raw materials like water, malt, etc are required. There are 10 different materials. The production of one crate of $b(j)$ requires q_{jk} units of material k $(k = 1, \ldots, 10)$. Materials are sold in batches; one batch of material k contains B_k units, and costs d_k euro. Extend the model of part (b) to include this option.

(d) For each material, the initial inventory equals 0. At the end of the production period, unused material has to be stored for the next period; for material k this costs v_k per unit. Extend the model of part (c) to include this option.

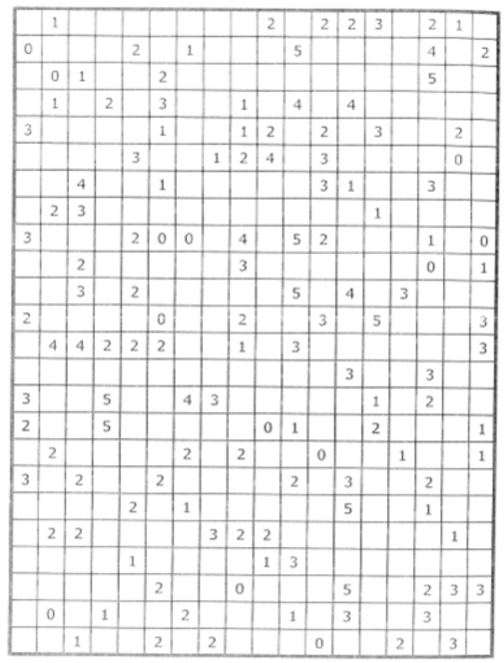

Figure 1: An example of treasure island

Additional challenge: Treasure Island with pitfall

We consider the puzzle called 'Treasure island'. On this island some diamonds are buried, and we have to find them. The island is modelled by a grid of m rows and n columns. In each grid point (i, j) at most one diamond may be buried. For a subset G of the positions, we are given a number g_{ij} which equals the total number of diamonds buried in the positions directly surrounding (i, j) (note that for a point in the interior of the grid these are 8 positions). There are no diamonds on the positions in G. An exampl e is depicted above. We consider Treasure island with pitfall. In this case *exactly one* of the numbers g_{ij} is wrong. Formulate an Integer Linear Programming problem that solves this variant of the puzzle. Note that this is in fact a feasibility problem, i.e. we want to decide if there is a feasible solution, so that we can choose the objective to be constant.

Additional challenge PRM

Formulate the problem of finding an optimal schedule for employees assisting PRMs at airports as an Integer Linear Programming problem. This is a difficult exercise where you have a lot of freedom. You can define the modelling assumptions by yourself. The advise is to take a stepwise approach. Start with a simplified version of the problem and then introduce additonal features one by one. A working paper on the PRM problem is available on the course website.