3.8 Strong valid inequalities for structured IP problems

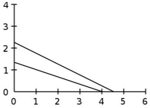
By studying the problem structure, we can derive strong valid inequalities which lead to better approximations of the ideal formulation $conv(X)$ and hence to tighter bounds.

Consider a polyhedron $P = \{\underline{x} \in \mathbb{R}^n_+ \; : \; A\underline{x} \leq \underline{b}\}$.

Definition: Given two valid inequalities $\underline{\pi}^t\underline{x} \leq \pi_0$ and $\underline{\mu}^t\underline{x} \leq \mu_0$ for P, $\underline{\pi}^t\underline{x} \leq \pi_0$ **dominates** $\underline{\mu}^t\underline{x} \leq \mu_0$ if $\exists\ u > 0$ such that $u\underline{\mu} \leq \underline{\pi}$ and $\pi_0 \leq u\mu_0$ with $(\underline{\pi}, \pi_0) \neq (u\underline{\mu}, u\mu_0)$.

Since $u\underline{\mu}^t\underline{x} \leq \underline{\pi}^t\underline{x} \leq \pi_0 \leq u\mu_0$, clearly $\{\underline{x} \in \mathbb{R}^n_+ : \underline{\pi}^t\underline{x} \leq \pi_0\} \subseteq \{\underline{x} \in \mathbb{R}^n_+ : \underline{\mu}^t\underline{x} \leq \mu_0\}$.

Example: $x_1 + 3x_2 \leq 4$ dominates $2x_1 + 4x_2 \leq 9$ since for $(\underline{\pi}, \pi_0) = (1, 3, 4)$ and $(\underline{\mu}, \mu_0) = (2, 4, 9)$ we have $\frac{1}{2}\underline{\mu} \leq \underline{\pi}$ and $\pi_0 \leq \frac{1}{2}\mu_0$.

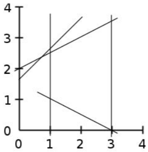


Definition: A valid inequality $\underline{\pi}^t \underline{x} \leq \pi_0$ is **redundant** in the description of P if there exist $k \geq 2$ valid inequalities $\underline{\pi}^i \underline{x} \leq \pi_0^i$ for P with $u_i > 0$, $1 \leq i \leq k$, such that

$$(\sum_{i=1}^{k} u_i \underline{\pi}^i)\underline{x} \leq \sum_{i=1}^{k} u_i \pi_0^i \quad \text{dominates} \quad \underline{\pi}^t \underline{x} \leq \pi_0.$$

Example:

$$P = \{(x_1, x_2) \in \mathbb{R}^2_+ \ : \ -x_1 + 2x_2 \leq 4, \ -x_1 - 2x_2 \leq -3, \ -x_1 + x_2 \leq 5/3, \ 1 \leq x_1 \leq 3\}$$



$-x_1 + x_2 \leq 5/3$ is redundant because it is dominated by $-x_1 + x_2 \leq 3/2$, which is implied by $-x_1 + 2x_2 \leq 4$ and $-x_1 \leq -1$ (with $u_1 = u_2 = \frac{1}{2}$)

Observation: When $P = conv(X)$ is not known explicitly it can be very difficult to check redundancy. In pratice, we should avoid inequalities that are dominated by others

3.8.1 Faces and facets of polyhedra

Consider a polyhedron $P = \{\underline{x} \in \mathbb{R}^n \ : \ A\underline{x} \leq \underline{b}\}$

Definitions

- The vectors $\underline{x}_1, \ldots, \underline{x}_k \in \mathbb{R}^n$ are **affinely independent** if the $k-1$ vectors $\underline{x}_2 - \underline{x}_1, \ldots, \underline{x}_k - \underline{x}_1 \in \mathbb{R}^n$ are linearly independent, or equivalently if the k vectors $(\underline{x}_1, 1), \ldots, (\underline{x}_k, 1) \in \mathbb{R}^{n+1}$ are linearly independent.
- The **dimension** of P, $\dim(P)$, is equal to the maximum number of affinely linearly independent points of P minus 1.
- P is **full-dimensional** if $\dim(P) = n$, i.e., no equation $\underline{a}^t \underline{x} = b$ is satisfied with equality by all the points $\underline{x} \in P$.

Illustrations:

For the sake of simplicity, we assume that P is full dimensional

Theorem: If P is of full dimension, P admits a unique minimal description

$$P = \{\underline{x} \in \mathbb{R}^n : \underline{a}_i^t \underline{x} \leq b_i, i = 1, \ldots, m\}$$

where each inequality is unique within a positive multiple.

Each inequality is necessary: deleting anyone of them we obtain a polyhedron that differs from P.

Moreover, each valid inequality for P which is not a positive multiple of one of the $\underline{a}_i^t \underline{x} \leq b_i$ is redundant (can be obtained as linear combination with nonnegative coefficients of two or more valid inequalities).

Alternative characterization of necessary valid inequalities

Definitions

- Let $F = \{\underline{x} \in P : \underline{\pi}^t \underline{x} = \pi_0\}$ for any valid inequality $\underline{\pi}^t \underline{x} \leq \pi_0$ for P. Then F is a **face** of P and the inequality $\underline{\pi}^t \underline{x} \leq \pi_0$ *represents* or *defines* F.
- If F is a face of P and $\dim(F) = \dim(P) - 1$, then F is a **facet** of P.

Illustrations:

Consequences: The faces of a polyhedron are polyhedra and a polyhedron has a finite number of faces.

Theorem: If P is full dimensional, a valid inequality is necessary for the description of P if and only if it defines a facet of P, i.e., if there exist n affinely independent points of P satisfying it at equality.

Example

Consider the polyhedron $P \subset \mathbb{R}^2$ described by:

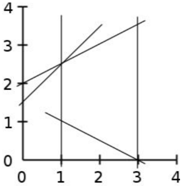
$$-x_1 + 2x_2 \leq 4 \quad (1)$$

$$-x_1 - 2x_2 \leq -3 \quad (2)$$

$$-x_1 + x_2 \leq \frac{3}{2} \quad (3)$$

$$x_1 \leq 3 \quad (4)$$

$$x_1 \geq 1 \quad (5)$$



Verify that P is full dimensional (dim(P=2).

Which inequalities define facets of P or are redundant? All but (3) define facets.

How to show that a valid inequality is facet defining

Consider $X \subset Z^n_+$ and a valid inequality $\underline{\pi}^t \underline{x} \leq \pi_0$ for X

Assumption: $conv(X)$ is bounded and full dimensional

Two simple approaches to show that $\underline{\pi}^t \underline{x} \leq \pi_0$ defines a facet of $P = conv(X)$:

1) Apply the definition: Find n points $\underline{x}^1, \ldots, \underline{x}^n \in X$ that satisfy the inequality with equality ($\underline{\pi}^t \underline{x} = \pi_0$) and are affinely independent.

2) Indirect approach:

(i) Select t points $\underline{x}^1, \ldots, \underline{x}^t \in X$, with $t \geq n$, that satisfy $\underline{\pi}^t\, \underline{x} = \pi_0$. Suppose that they all belong to a generic hyperplane $\underline{\mu}^t \underline{x} = \mu_0$.

(ii) Solve the linear system

$$\sum_{j=1}^{n} \mu_j x_j^k = \mu_0 \qquad \text{for } k = 1, \ldots, t$$

in the $n + 1$ unknowns $\mu_0, \mu_1, \ldots, \mu_n$.

(iii) If the only solution is $(\underline{\mu}, \mu_0) = \lambda(\underline{\pi}, \pi_0)$ with $\lambda \neq 0$, then the inequality $\underline{\pi}^t\, \underline{x} \leq \pi_0$ defines a facet of $conv(X)$.

Example:

Consider $X = \{(\underline{x}, y) \in \mathbb{R}^m \times \{0,1\} \,:\, \sum_{i=1}^{m} x_i \leq my,\ 0 \leq x_i \leq 1\ \forall i\}$

i) Verify that $dim(conv(X)) = m+1$. $(\underline{0},0)$, $(\underline{0},1)$ and $(\underline{e}_i,1)$, with $1 \leq i \leq m$, are $m+2$ affinely independent points of $conv(X)$.

ii) Show with indirect approach that for each i the valid inequality $x_i \leq y$ defines a facet of $conv(X)$.

Consider the ~~$2m$~~ $m+1$ points $(\underline{0},0)$, $(\underline{e}_i,1)$ and $(\underline{e}_i + \underline{e}_{i'},1)$ for $i' \neq i$, which are feasible and satisfy $x_i = y$.

Since $(\underline{0},0)$ belongs to the hyperplane defined by $\sum_{j=1}^{m} \mu_j x_j + \mu_{m+1} y = \mu_0$, then $\mu_0 = 0$.

Since $(\underline{e}_i,1)$ belongs to the hyperplane defined by $\sum_{j=1}^{m} \mu_j x_j + \mu_{m+1} y = \mu_0 = 0$, then $\mu_i = -\mu_{m+1}$.

Since $(\underline{e}_i + \underline{e}_{i'},1)$ belongs to the hyperplane defined by $\sum_{j=1}^{m} \mu_j x_j - \mu_i y = \mu_0 = 0$, then $\mu_{i'} = 0$ for $i' \neq i$.

Thus the hyperplane is $\mu_i x_i - \mu_i y = 0$ and hence $x_i \leq y$ defines a facet of $conv(X)$.

3.8.2 Cover inequalities for the binary knapsack problem

Consider $X = \{\underline{x} \in \{0,1\}^n : \sum_{j=1}^{n} a_j x_j \leq b\}$ with $b > 0$ and $N = \{1, \ldots, n\}$.

Assumptions: For each j with $1 \leq j \leq n$, $a_j > 0$ (if $a_j < 0$ set $x_j' = 1 - x_j$) and $a_j \leq b$.

Definitions: A subset $C \subseteq N$ is a **cover** for X if $\sum_{j \in C} a_j > b$.
A cover is **minimal** if, for each $j \in C$, $C \setminus \{j\}$ is not a cover.

Example: For $X = \{\underline{x} \in \{0,1\}^7 : 11x_1 + 6x_2 + 6x_3 + 5x_4 + 5x_5 + 4x_6 + x_7 \leq 19\}$
$\{1,2,3\}$ is a minimal cover and $\{3,4,5,6,7\}$ is a non-minimal cover

Proposition: If $C \subseteq N$ is a cover for X, the inequality

$$\sum_{j \in C} x_j \leq |C| - 1$$

is valid for X, and is called a **cover inequality**.

Example cont.: for above covers $x_1 + x_2 + x_3 \leq 2$ and $x_3 + x_4 + x_5 + x_6 + x_7 \leq 3$

Proposition: Let $C \subseteq N$ be a cover for X. The cover inequality associated to C

$$\sum_{j \in C} x_j \leq |C| - 1$$

defines a facet of $P_C := conv(X) \cap \{\underline{x} \in \mathbb{R}^n : \; x_j = 0, j \in N \setminus C\}$ if and only if C is a minimal cover.

Separation of cover inequalities

Separation problem: Given a fractional solution $\underline{x}^*$ with $0 \leq x_j^* \leq 1$, $1 \leq j \leq n$, decide whether $\underline{x}^*$ satisfies all the cover inequalities or determine one violated by $\underline{x}^*$.

Since $\sum_{j \in C} x_j \leq |C| - 1$ can be written as $\sum_{j \in C} (1 - x_j) \geq 1$, this amounts to answer the question:

Does there exist a subset $C \subseteq N$ such that $\sum_{j \in C} a_j > b$ and $\sum_{j \in C} (1 - x_j^*) < 1$?

If $\underline{z} \in \{0, 1\}^n$ is the incidence vector (binary characteristic vector) of the subset $C \subseteq N$, it is equivalent to the question:

$$\zeta^* = \min\{\sum_{j \in N} (1 - x_j^*) z_j : \sum_{j \in N} a_j z_j > b, \underline{z} \in \{0, 1\}^n\} < 1?$$

Proposition:

(i) If $\zeta^* \geq 1$, $\underline{x}^*$ satisfies all the cover inequalities.

(ii) If $\zeta^* < 1$ with optimal solution $\underline{z}^*$, then $\sum_{j \in C} x_j \leq |C| - 1$ with $C = \{j : z_j^* = 1, 1 \leq j \leq n\}$ cuts (is violated by) $\underline{x}^*$ by a quantity $1 - \zeta^*$.

Example:

Consider

$$\begin{aligned} \max \quad & z = 5x_1 + 2x_2 + x_3 + 8x_4 \\ s.t. \quad & 4x_1 + 2x_2 + 2x_3 + 3x_4 \leq 4 \\ & x_1, x_2, x_3, x_4 \in \{0, 1\} \end{aligned}$$

Optimal solution of the LP relaxation $\underline{x}^*_{LP} = (1/4, 0, 0, 1)$ with $z^*_{LP} = 9.25$.

The separation problem amounts to the following binary knapsack problem:

$$\begin{aligned} \zeta^* = \min \quad & \tfrac{3}{4} z_1 + z_2 + z_3 \\ s.t. \quad & 4z_1 + 2z_2 + 2z_3 + 3z_4 > 4 \\ & z_1, z_2, z_3, z_4 \in \{0, 1\} \end{aligned}$$

where the $>$ constraint can be replaced with $4z_1 + 2z_2 + 2z_3 + 3z_4 \geq 5$.

Optimal solution $\underline{z} = (1, 0, 0, 1)$ with $\zeta^* = \frac{3}{4}$.

Thus the cover inequality

$$x_1 + x_4 \leq 1$$

cuts away the current LP optimal solution $\underline{x}^*_{LP}$ by $1 - \zeta^* = \frac{1}{4}$.

Can such cover inequalities be strengthened?

Proposition: If $C \subseteq N$ is a cover for X, the **extended cover inequality**

$$\sum_{j \in E(C)} x_j \leq |C| - 1$$

is valid for X, where $E(C) = C \cup \{j \in N \, : \, a_j \geq a_i \text{ for all } i \in C\}$.

Example cont.: For $X = \{\underline{x} \in \{0,1\}^7 : 11x_1 + 6x_2 + 6x_3 + 5x_4 + 5x_5 + 4x_6 + x_7 \leq 19\}$, the extended cover inequality for $C = \{3,4,5,6\}$ is

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 3$$

which clearly dominates

$$x_3 + x_4 + x_5 + x_6 \leq 3. \tag{6}$$

Observation: Since $a_1 = 11$, $a_i \geq 5$ for $i \in \{3,4,5\}$, $a_6 = 4$ and $b = 19$, if $x_1 = 1$ at most one of the other variables in (6) can take value 1 and the inequality

$$2x_1 + x_3 + x_4 + x_5 + x_6 \leq 3$$

is valid and in turn dominates (6).

How can we systematically strengthen a cover inequality $\sum_{j \in C} x_j \leq |C| - 1$ to obtain a facet defining one?

Lifting procedure:

Let $j_1, \ldots, j_r$ be the indices of $N \setminus C$.

Iteration 1: Determine the maximum value of α_{j_1} such that

$$\alpha_{j_1} x_{j_1} + \sum_{j \in C} x_j \leq |C| - 1$$

is valid for X by solving the (binary knapsack) problem

$$\begin{array}{rl} \sigma_1 = \max & \sum_{j \in C} x_j \\ s.t. & \sum_{j \in C} a_j x_j \leq b - a_{j_1} \\ & \underline{x} \in \{0,1\}^{|C|} \end{array}$$

and by setting $\alpha_{j_1} = |C| - 1 - \sigma_1$.

$\sigma_1 =$ maximum amount of "space" used up by the variables of indices in C when $x_{j_1} = 1$.

Iteration 2: Determine the maximum value of α_{j_2} such that

$$\alpha_{j_2} x_{j_2} + \alpha_{j_1} x_{j_1} + \sum_{j \in C} x_j \leq |C| - 1$$

is valid for X by solving the (binary knapsack) problem

$$\begin{array}{rl} \sigma_2 = \max & \alpha_{j_1} x_{j_1} + \sum_{j \in C} x_j \\ s.t. & a_{j_1} x_{j_1} + \sum_{j \in C} a_j x_j \leq b - a_{j_2} \\ & \underline{x} \in \{0,1\}^{|C|+1} \end{array}$$

and by setting $\alpha_{j_2} = |C| - 1 - \sigma_2$.

Iteration 3: ...

Example cont.:

For $X = \{\underline{x} \in \{0,1\}^7 : 11x_1 + 6x_2 + 6x_3 + 5x_4 + 5x_5 + 4x_6 + x_7 \leq 19\}$,
applying the lifting procedure to

$$x_3 + x_4 + x_5 + x_6 \leq 3$$

considering in the order x_1, x_2 and x_7, we obtain the valid inequality

$$2x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 3.$$

Lifting procedure for cover inequalities

Let $j_1, \dots, j_r$ be the indices of $N \setminus C$ and set $t = 1$.

Let $\sum_{i=1}^{t-1} \alpha_{j_i} x_{j_i} + \sum_{j \in C} x_j \leq |C| - 1$ be the inequality obtained at iteration $t-1$.

Iteration t: Determine the maximum value of α_{j_t} such that

$$\alpha_{j_t} x_{j_t} + \sum_{i=1}^{t-1} \alpha_{j_i} x_{j_i} + \sum_{j \in C} x_j \leq |C| - 1$$

is valid for X by solving the (binary knapsack) problem

$$\begin{aligned} \sigma_t = \max \quad & \textstyle\sum_{i=1}^{t-1} \alpha_{j_i} x_{j_i} + \sum_{j \in C} x_j \\ s.t. \quad & \textstyle\sum_{i=1}^{t-1} a_{j_i} x_{j_i} + \sum_{j \in C} a_j x_j \leq b - a_{j_t} \\ & \underline{x} \in \{0,1\}^{|C|+t-1} \end{aligned}$$

and by setting $\alpha_t = |C| - 1 - \sigma_t$.

Terminate when $t = r$.

Note: $\sigma_t =$ maximum amount of "space" used up by the variables of indices in $C \cup \{j_1, \dots, j_{t-1}\}$ when $x_{j_t} = 1$.

Proposition: If $C \subseteq N$ is a minimal cover and $a_j \leq b$ for all $j \in N$, the lifting procedure is guaranteed to yield a facet defining inequality of $conv(X)$.

Example cont.:

For $X = \{\underline{x} \in \{0,1\}^7 : 11x_1 + 6x_2 + 6x_3 + 5x_4 + 5x_5 + 4x_6 + x_7 \leq 19\}$, the valid inequality

$$2x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 3$$

defines a facet of $conv(X)$.

Clearly, the resulting facet defining inequality depends on the order in which the variables of $N \setminus C$ are considered, that is, on the lifting sequence.

3.8.3 Facets of the traveling salesman problem

STSP: Given an undirected graph $G = (V, E)$ with $n = |V|$ nodes and a cost c_e for every edge $e = \{i, j\} \in E$, determine a Hamiltonian cycle of G of minimal total cost.

$$\begin{array}{llr}
\min & \sum_{e\in E} c_e x_e & \\
\text{s.t.} & \sum_{e\in\delta(i)} x_e = 2 & i \in V \\
& \sum_{e\in E(S)} x_e \leq |S| - 1 & S \subset V,\ S \neq \emptyset \\
& x_e \in \{0, 1\} & e \in E.
\end{array}$$

Let X denote the set of all incidence vectors $\underline{x} \in \{0, 1\}^{|E|}$ of Hamiltonian cycles.

Proposition: For every $S \subseteq V$ with $2 \leq |S| \leq n/2$ and $n \geq 4$,

$$\sum_{e\in E(S)} x_e \leq |S| - 1 \qquad (7)$$

defines a facet of the STSP polytope $conv(X)$.

The STSP polytope $conv(X)$ has a very complicated structure. Many classes of facet defining inequalities are known but its complete description is unknown.

3.8.4 Equivalence between separation and optimization

Consider a family of LPs $\min\{\, \underline{c}^t \underline{x} \, : \, \underline{x} \in P_o \}$ with $o \in \mathcal{O}$, where $P_o = \{\, \underline{x} \in \mathbb{R}^{n_o} \, : \, A_o \underline{x} \geq \underline{b}_o \,\}$ polytope with rational (integer) coefficients and a very large (e.g., exponential) number of constraints.

Examples:

1) Linear relaxation of asymmetric TSP with cut-set inequalities ($\mathcal{O}$ set of all graphs)

2) Maximum Matching problem: For each $G = (V, E)$, the matching polytope

$$conv(\{\underline{x} \in \{0,1\}^{|E|} \, : \, \sum_{e \in \delta(i)} x_e \leq 1, \, \forall i \in V \,\})$$

coincides (Edmonds) with

$$\{\underline{x} \in \mathbb{R}_+^{|E|} \, : \, \sum_{e \in \delta(i)} x_e \leq 1, \, \forall i \in V, \, \sum_{e \in E(S)} x_e \leq \frac{|S| - 1}{2}, \, \forall S \subseteq V \text{ with } |S| \geq 3 \text{ odd}\}$$

Consider a cutting plane approach where constraints are only generated if needed.

Assumption: Even though the number of constraints m_o of P_o is exponential in n_o (e.g., $O(2^{n_o})$), A_o and $\underline{b}_o$ are specified in a concise way (as a function of a polynomial number of parameters w.r.t. n_o).

Equivalence between optimization and separation

Optimization problem: Given a rational polytope $P \subseteq \mathbb{R}^n$ and a rational objective vector $\underline{c} \in \mathbb{R}^n$, find a $\underline{x}^* \in P$ minimizing $\underline{c}^t \underline{x}$ over $\underline{x} \in P$ or establish that P is empty.

N.B.: we assume that P is bounded (polytope) just to avoid unbounded problems.

Separation problem: Given a rational polytope $P \subseteq \mathbb{R}^n$ and a rational vector $\underline{x}' \in \mathbb{R}^n$, establish that $\underline{x}' \in P$ or determine a cut that separates $\underline{x}'$ from P (a rational vector $\underline{\pi} \in \mathbb{R}^n$ such that $\underline{\pi x} < \underline{\pi x}'$ for each $\underline{x} \in P$)

Theorem: (consequence of Grötschel, Lovász, Schriver 1988 theorem)

The separation problem for a family of polyhedra can be solved in polynomial time in n and $\log U$ if and only if the optimization for that family can be solved in polynomial time in n and $\log U$, where U is an upper bound on all a_{ij} and b_i.

Proof based on *Ellipsoid method*, first polynomial algorithm for LP (Khachiyan 1979).

For now it is a theoretical tool: the resulting algorithm is not efficient but the equivalence may guide the search for more practical polynomial-time algorithms.

Corollary: The linear relaxation of the ILP formulation for ATSP with cut-set inequalities can be solved in polynomial time in spite of the exponential number of constraints.

3.8.5 Remarks on cutting plane methods

Consider a generic discrete optimization problem

$$\min\{ \underline{c}^t \underline{x} \; : \; \underline{x} \in X \subseteq \mathbb{R}^n_+ \}$$

with rational coefficients c_i.

When designing a cutting plane method, be aware that:

- It can be difficult to describe one or more families of strong (possibly facet defining) valid inequalities for $conv(X)$.
- The separation problem for a given family $\mathcal{F}$ may require a considerable computational effort (if NP-hard devise heuristics).
- Even when finite convergence is guaranteed (e.g., with Gomory cuts), pure cutting plane methods tend to be very slow.

The subfield of Discrete Optimization studying the polyhedral structure of the ideal formulations ($conv(X)$) is known as Polyhedral Combinatorics.